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On Fuzzy Soft (n, m) Binormal Operator

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Abstract

A new type of operators in fuzzy soft Hilbert spaces is presented and explored in this paper, which is the fuzzy soft (n,m) binormal operator. Expanding the classical concept of binormal operators to the principles of the fuzzy soft sets, we determine the essential nature of this type of operators and investigate the main characteristics of the operators. A number of theorems that explain the structural property of fuzzy soft (n,m) binormal operators are demonstrated such as a result on adjoint operators, scalar multiplication, and commutativity. In addition, we discuss the interrelationships amongst this new defined operator with other already existing classes of fuzzy soft linear operators, specifically fuzzy soft self-adjoint operator and fuzzy soft normal operator. It is proved that the (n,m) binormal property of operator products, direct sums and the tensor products, when applied under the right conditions of commuting, as well as preserving the property. The study is part of the current advancement of the operator theory in fuzzy soft mathematical framework, offering the fundamental outcomes to the continued research in the fuzzy soft functional analysis.

Keywords: Fuzzy Soft Sets; Fuzzy Soft Hilbert Space; Fuzzy Soft Linear Operator; Fuzzy Soft Binormal Operator

1. Introduction

In order to deal with insecurity, Zadeh [1] proposed an extension of the theory of sets, which is the concept of fuzzy sets, in 1965. A membership function from domain Y to 0.1 is a fuzzy set on domain Y. Molodtsov [2] suggested new kinds of sets in 1999, which are commonly referred to as soft sets. By linking a collection to a set of parameters, the soft set a parameterized family of Universal Set sub sets is a mathematical tool for representing uncertainty.

Following that, other scholars have proposed enlarged, new concepts based on soft sets. They supplied examples and studied their properties, including soft point [3] and soft normal space [4]. Soft Hilbert space [5,6] and soft Inner Product Space [5].

Many researchers, like Maji [7], combine fuzzy and soft sets for producing a fuzzy soft set. A number of ideas, Fuzzy soft normed space, and fuzzy soft point [8]. It has been introduced using this approach [9]. Presently, Faried et al. described the fuzzy soft Hilbert space and fuzzy soft inner product space [10]. The operator of fuzzy soft linear [11] is another. At last, they stare at the qualities of the fuzzy soft self-adjoint operator [12]. The indicator of this inquiry is to propose a fuzzy soft (n,m) binormal operator, a new kind of binormal operator, and provide certain theorems and properties associated with this operator, as well as how this operator relates to other types. Like the self-adjoint fuzzy soft operator.

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2. Essential concepts

2.1. Definition [1]

It is assumed that the fuzzy set $\hat{A}$ has a membership function over the universe X. It is represented by a sequence of ordered pairings: $\hat{A} = \{(x, \mu_{\hat{A}}(\chi)) | x \in \chi, \mu_{\hat{A}} \in \Im \}$, $\hat{}\mu_{\hat{A}}: \chi \longrightarrow \Im, where\ \Im = [0,1]\ \ and\ \hat{A}\ \ or\ \hat{A} = \{\frac{\mu_{\hat{A}}(\chi)}{x}: x \in X$

The degree of membership of x in $\hat{A}$ is denoted by $\mu_{\hat{A}}(x)$. Additionally, a function from X into $\Im$ is $\Im^X$={ $\hat{A}$.

2.2. Definition [2]

Let A⊆E, where Ł (X) is the force coalition of X, E is a set of exchanges, and X is a universal system. Given that g is a mapping, it offers: A ⟶ Ł (X), where

$$g_A = \{g(e) \in Ł(X): e \in A\}.$$

Gentle set over X with regard to A is the pair (g, A) or g_ A.

2.3. Definition [7]

The mysterious soft assembly (FS-set) over a universe system X is defined as a soft assembly (g, A). If g is a mapping, then $g: A \longrightarrow \Im^X$, and $\{g(e) \in: e \in A\}$ represents the set of elements obtained by applying the mapping g to the elements of A.

$FSS(\ \hat{X})$ is the symbol for the entire family of FS-sets.

2.4. Definition [8]

The FS-set, denoted (g, A). FSS ($\hat{X}$). Over a set X, an unclear soft point

where e ∈ A and x ∈ X, is represented by a $\left(\left(X_{\mu_{g(e)}}, A\right)\ or\ \hat{X}_{\mu_{g(e)}}\right)$ where e ∈ A.

The definition of the function $\mu_{g(e)}(X)$is as follows:

$$\mu_{g(e)}(X) = \begin{cases} \lambda & , if\ x = x_{-0}\ \in X\ and\ e = e_{-0}\ \in A \\ 0 & , if\ x\ \in X - \{x_{-0}\}\ or\ e \in A - \{e_{-0}\} \end{cases},\quad where\ \lambda \in (0,1]$$

2.5. Remark [8]

All FS-complex numbers belong to the family C (A), and all FS-real numbers belong to the family R(A).

2.6. Definition [9]

$\hat{X}$ is a representation of the FS-vector space. Next, a mapping Fuzzy soft norm on $\hat{X}$ (FSN) is defined as: $\|.\|: \hat{X}\ \rightarrow R(A)$ if the following criteria are met:

1. $\left\| \hat{X}_{\mu_{g\,(e)}} \right\| \geq 0, \forall\ \hat{X}_{\mu_{g\,(e)}} \in \hat{X}\ , and\ \left\| \hat{X}_{\mu_{g_{(e)}}} \right\| = \hat{0}\ \Leftrightarrow\ \ \hat{X}_{\mu_{g_{(e)}}} = \hat{0}.$
2. $\left\| \hat{Ł}\, \hat{X}_{\mu_{g\,(e)}} \right\| = \ |\hat{Ł}|\ .\ \hat{X}_{\mu_{g\,(e)}},\ \ \hat{X}_{\mu_{g\,(e)}} \forall\ \hat{X}, \hat{Ł} \in C(A).$
3. $\left\| \hat{X}_{\mu_{1_{g\,(e_1)}}} +\ \hat{Y}_{\mu_{2_{g\,(e_2)}}} \right\| \leq\ \hat{X}_{\mu_{1_{g\,(e_1)}}} +\ \hat{Y}_{\mu_{2_{g\,(e_2)}}}, \forall\ \hat{X}_{\mu_{1_{g\,(e_1)}}},\ \hat{Y}_{\mu_{2_{g\,(e_2)}}} \forall\ \hat{X}.$

The FS-vector space with FSN. Fuzzy soft normed vector space, or FSN-space. It is represented by the symbol $\left(\hat{X}, \|.\|\right)$.

2.7. Definition [10]

Let $\hat{X}$ represent the FSV-space. After that, mapping takes place. If ⟨...,.⟩ satisfies the following criteria, then $\hat{X}\ \times \hat{X} \rightarrow$C(A) or R(A) is referred to as the fluffy. On $\hat{X}$, the soft inner product.

(FSV).

1. $\langle \hat{x}_1, \hat{x}_2 \rangle \geq \hat{0}, \quad \forall \hat{x}_i \in \hat{X} \Leftrightarrow \hat{0} \in \hat{X}$
2. $\langle \hat{x}_1, \hat{e}_2 \rangle * \langle \hat{x}_2, \hat{e}_1 \rangle = \hat{\alpha} \langle \hat{x}_1, \hat{e}_2 \rangle * \langle \hat{x}_2, \hat{e}_3 \rangle \forall \alpha$

The space of FS-vectors $FSV\langle .,. \rangle$is known as $(\hat{X}, \langle .,. \rangle)$. represents the fuzzy soft product space (FSV-space).

2.8. Definition [13]

Fuzzy soft convergent is a series of FS-vectors $\{\hat{X}^n{}_{\mu_{n_{g(e_n)}}}\}$ in FSN-space $(\hat{X}, \|.\|)$ that converges to $\hat{X}^o{}_{\mu_{0_{g\,(e_0)}}}$ if $\lim_{n\to\infty} \left\| \hat{X}^n{}_{\mu_{n_{g(e_n)}}} - \hat{X}_{\mu_{0_{g\,(e_0)}}} \right\| = \hat{0}$, $i.e. \forall \hat{\in} > \hat{0}, \exists n_0 \in N$ such as that $\left\| \hat{X}^n{}_{\mu_{n_{g(e_n)}}} - \hat{X}^o{}_{\mu_{0_{g\,(e_0)}}} \right\| < \hat{\in}, \quad \forall n \geq n_0$.

It if denoted by $n \to \infty$ $as\ \hat{X}^n{}_{\mu_{n_{g(e_n)}}} \to \hat{X}^o{}_{\mu_{0_{g\,(e_0)}}}$ or . $\lim_{n\to\infty} \hat{X}^n{}_{\mu_{n_{g(e_n)}}} = \hat{X}^o{}_{\mu_{0_{g\,(e_0)}}}$

2.9. Definition [13]

A series of FS-vectors in FSN-space $(\hat{X}, \|.\|)$ $\{\hat{X}^n{}_{\mu_{n_{g\,(e_n)}}}\}$. Fuzzy soft Cauchy sequence is the term for it. if $\forall \hat{\in} > \hat{0}\ \exists n_0 \in N$ such as that $\left\| \hat{X}^n{}_{\mu_{n_{g\,(e_n)}}} - \hat{X}^m{}_{\mu_{m_{g\,(e_m)}}} \right\| < \hat{\in}, \quad \forall n, m \geq, m < n.$

To put it another way $\left\| \hat{X}^n{}_{\mu_{n_{g(e_n)}}} - \hat{X}^m{}_{\mu_{m_{g\,(e_m)}}} \right\| \Rightarrow \hat{0}\ as\ n, m \to \infty.$

2.10. Definition [13]

The FSN-space $(\hat{X}, \|.\|)$ Is called fuzzy soft complete (FS-complete) if every FS-Cauchy sequence is an FS-convergence sequence in it.

2.11. Definition [10]

The FS-space $(\hat{X}, \langle .,. \rangle)$ if is FS-complete in the induced FSN

$\left\| \hat{X}_{\mu_{g\,(e)}} \right\| = \sqrt{\langle \hat{X}_{\mu_{g\,(e)}}, \hat{X}_{\mu_{g\,(e)}} \rangle}$ is represented by the symbol $(\widehat{H}, \langle .,. \rangle)$, and is known as a fuzzy soft Hilbert space (FSH-space).

2.12. Definition [11]

Fuzzy soft linear operators (FSL-operators) are defined as follows: Let $\widehat{H}$ be an FSH-space, and $T: \widehat{H} \to \widehat{H}$ be a gentle fuzzy operator.

1. $\hat{Ł}\left(\hat{X}_{\mu_{1_{g\,(e_1)}}} . \hat{Y}_{\mu_{2_{g\,(e_2)}}} \right) = \hat{Ł}\left(\hat{X}_{\mu_{2_{g\,(e_1)}}} \right) . \hat{Ł}\left(\hat{Y}_{\mu_{2_{g\,(e_2)}}} \right) \ \forall \hat{X}_{\mu_{1_{g\,(e_1)}}} . \hat{Y}_{\mu_{2_{g\,(e_2)}}} \in \widehat{H}$
2. $\hat{Ł}\left(\hat{\beta} * \hat{X}_{\mu_{1_{g\,(e_1)}}} \right) = \hat{\beta} * \hat{Ł}\left(\hat{X}_{\mu_{1_{g\,(e_1)}}} \right), \ \forall \hat{X}_{\mu_{1_{g\,(e_1)}}} \in \widehat{H}, and\ \hat{\beta} \in C(A)$

i.e. $\hat{Ł}\left(\hat{\alpha} \widehat{* X}_{\mu_{1_{g\,(e_1)}}} + \hat{\beta} * \hat{Y}_{\mu_{1_{g\,(e_1)}}} \right) = \hat{\alpha} \hat{*} \hat{Ł}\left(\hat{X}_{\mu_{1_{g\,(e_1)}}} \right) + \hat{\beta} * \hat{Ł}\left(\hat{Y}_{\mu_{2_{g\,(e_2)}}} \right) \ \forall \hat{X}_{\mu_{1_{g\,(e_1)}}}, \hat{Y}_{\mu_{2_{g\,(e_2)}}} \in \widehat{H}$ and $\hat{\alpha}, \hat{\beta}$ any fuzzy soft scalars.

2.13. Definition [11]

The FSB-operator, or fuzzy soft bounded operator is defined as follows: if $\exists\ \widehat{m} \in R(A)$ then let $\widehat{H}$ be the FSH-area, and $\hat{Ł}: \widehat{H} \to \widehat{H}$ be the gentle fuzzy operator.

$$\hat{Ł}\, \hat{X}_{\mu_{1_{g\,(e_1)}}} \leq \widehat{m}\, \hat{X}_{\mu_{1_{g\,(e_1)}}}, \text{ for all } \hat{X}_{\mu_{1_{g\,(e_1)}}} \in \widehat{H}.$$

That set from bounded and FS-linear agents is now represented as $\hat{B}\left(\widehat{H}\right)$.

2.14. Definition [11]

Let $\widehat{H}$ and $\widehat{K}$ be two FSH areas. The scope of work of the FS-operator $\hat{Ł} : \widehat{H} \to \widehat{K}$, denoted by Ram $\hat{Ł}$ and

$$\text{Rang } \hat{Ł} = \{\hat{Ł}\,(\hat{X}_{\mu_{1_{g\,(e_1)}}}) \in \widehat{K} : \hat{X}_{\mu_{1_{g\,(e_1)}}} \in \widehat{H}\},$$

The FS-operator $\hat{Ł} : \widehat{H} \to \widehat{K}$ null (kernel) is represented by Ker$(\widehat{Ł})$ and Ker$(\widehat{Ł})$=The equation

$$\text{Ker}(\widehat{Ł}) = \left\{\hat{X}_{\mu_{1_{g\,(e_1)}}} \in \widehat{H}: \hat{Ł}^{*}\hat{X}_{\mu_{1_{g\,(e_1)}}}\right\} = \hat{0}$$

2.15. Definition [11]

$\hat{I}\left(\hat{X}_{\mu_{1_{g\,(e_1)}}}\right) = \hat{x}_{\mu_{1_{g\,(e_1)}}}, \ \forall \hat{X}_{\mu_{1_{g\,(e_1)}}} \in \widehat{H}$. which is the FS-operator$\widehat{I} : \widehat{H} \to \widehat{H}$. It is also known as the fuzzy soft identity operator.

2.16. Definition [11]

Given an FSH-space$\widehat{H}$ and an FS-operator $\hat{Ł}: \widehat{H} \to \widehat{H}$, the literal meaning of $\hat{Ł}^{*}$, the fuzzy soft adjoint operator, is $\langle \hat{Ł}\hat{X}_{\mu_{1_{g(e_1)}}}, \hat{y}_{\mu_{2_{g(e_2)}}} \rangle = \langle \hat{X}_{\mu_{1_{g(e_1)}}}, \ \hat{Ł}^{*}\, \hat{y}_{\mu_{2_{g(e_2)}}} \rangle$

$$\forall\, \hat{X}_{\mu_{1_{g\,(e_1)}}}, \hat{y}_{\mu_{2_{g\,(e_2)}}} \in \widehat{H}.$$

2.17. Theorem [11]

Grant $\hat{Ł}, \hat{R} \in \hat{B}\left(\widehat{H}\right)$, also $\hat{\beta} \in C(A)$, then $(\hat{Ł}^{*})^{*} = \hat{Ł}$ where $\widehat{H}$ is FSH-space

$$(\hat{\beta}\hat{Ł})^{*} = \overline{\hat{\beta}}\hat{Ł}^{*}, (\hat{Ł} + \hat{R})^{*} = \hat{Ł}^{*} + \hat{R}^{*} \ and \ (\hat{Ł}\hat{R})^{*} = \hat{R}^{*}\hat{Ł}^{*}.$$

2.18. Theorem [11]

$\left\|\hat{Ł}^{*}\right\| = \left\|\hat{Ł}\right\| \ and \ \left\|\hat{Ł}^{*}\hat{Ł}\right\| = \left\|\hat{Ł}\right\|^{2}$. If $\hat{Ł} \in \hat{B}\ (\widehat{H}), where\ \widehat{H}$ is FSH-space.

2.19. Definition [12]

If $\hat{Ł} = \hat{Ł}^{*}$ then the FS-operator $\hat{Ł}$ of FSH-space $\widehat{H}$ is known as the fuzzy soft self-adjoint (FS-self-adjoint operator).

2.20. Definition [12-14]

If the $\hat{Ł}$ FS-operator on FSH-space $\widehat{H}$ can be reoesented as: $\hat{Ł} = \hat{R} + \hat{I}\hat{Ş}$

Where $\hat{Ł}\ and\ \hat{Ş}$ are operators over $\widehat{H}$ that are FS-self-adjoint

2.21. Definition [15]

Let $\hat{Ł}$ be the FS-operator of FSH-space $\widehat{H}$. If $\hat{Ł}\hat{Ł}^{*} - \hat{Ł}^{*}\hat{Ł} = 0$, then $\hat{Ł}$ is a FSN-operator is a fuzzy soft normal operator.

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3. Main results

3.1. Definition

If $\hat{Ł}$ is an FS-operator of FSH-space $\widehat{H}$ if $\hat{Ł}^{n}\hat{Ł}^{*^{m}}\hat{Ł}^{*^{m}}\hat{Ł}^{n} - \hat{Ł}^{*^{m}}\hat{Ł}^{n}\hat{Ł}^{n}\hat{Ł}^{*^{m}} = 0$, then $\hat{Ł}$ is a fuzzy soft (n,m) binormal operator $[FS - (\mathrm{n}, \mathrm{m}) - BN]$operator.

3.2. Remark

It is clear that every FS-self-adjoint operator is an $[FS - (\mathrm{n}, \mathrm{m}) - BN]$operator.

3.3. Theorem

Assume that $\hat{Ł} \in \hat{B}\left(\hat{H}\right)$ be an $[FS-(\mathrm{n,m})-BN]$operator then $\hat{Ł}^*$ is $[FS-(\mathrm{m,n})-BN]$operator.

3.3.1. Proof

Since $\hat{Ł}$ $[FS-(\mathrm{n,m})-BN]$operator then

$\hat{Ł}^n\hat{Ł}^{*^m}\hat{Ł}^{*^m}\hat{Ł}^n - \hat{Ł}^{*^m}\hat{Ł}^n\hat{Ł}^n\hat{Ł}^{*^m} = 0$, so that $\hat{Ł}^n\hat{Ł}^{*^m}\hat{Ł}^{*^m}\hat{Ł}^n = \hat{Ł}^{*^m}\hat{Ł}^n\hat{Ł}^n\hat{Ł}^{*^m}$

Since $\left(\hat{Ł}^*\right)^* = \hat{Ł}$,then

$$\left(\left(\hat{Ł}^*\right)^*\right)^n \hat{Ł}^{*^m}\hat{Ł}^{*^m}\left(\left(\hat{Ł}^*\right)^*\right)^n - \hat{Ł}^{*^m}\left(\left(\hat{Ł}^*\right)^*\right)^n\left(\left(\hat{Ł}^*\right)^*\right)^n\hat{Ł}^{*^m} = 0$$

Then $\hat{Ł}^*$is $[FS-(\mathrm{m,n})-BN]$ operator.

3.4. Proposition

Assume that $\hat{Ł}$ is $[FS-(\mathrm{n,m})-BN]$ operator then $K\hat{Ł}$ is $[FS-(\mathrm{n,m})-BN]$ operator for any $\in \mathbb{C}$. .

3.4.1. Proof

If $\hat{Ł}$ is $[IF-(\mathrm{n,m})-BN]$ operator then

$\hat{Ł}^n\hat{Ł}^{*^m}\hat{Ł}^{*^m}\hat{Ł}^n - \hat{Ł}^{*^m}\hat{Ł}^n\hat{Ł}^n\hat{Ł}^{*^m} = 0$, so that $\hat{Ł}^n\hat{Ł}^{*^m}\hat{Ł}^{*^m}\hat{Ł}^n = \hat{Ł}^{*^m}\hat{Ł}^n\hat{Ł}^n\hat{Ł}^{*^m}$

$$(K\hat{Ł})^n(K\hat{Ł})^{*^m}(K\hat{Ł})^{*^m}(K\hat{Ł})^n - (K\hat{Ł})^{*^m}(K\hat{Ł})^n(K\hat{Ł})^n(K\hat{Ł})^{*^m}$$

$$(K\hat{Ł})^n(K\hat{Ł})^{*^m}(K\hat{Ł})^{*^m}(K\hat{Ł})^n - \overline{K}^{*^m}\hat{Ł}^{*^m}K^n\hat{Ł}^n K^n\hat{Ł}^n \overline{K}^{*^m}\hat{Ł}^{*^m}$$

$(K\hat{Ł})^n(K\hat{Ł})^{*^m}(K\hat{Ł})^{*^m}(K\hat{Ł})^n - \overline{K}^{*^m}K^nK^n\overline{K}^{*^m}\hat{Ł}^{*^m}\hat{Ł}^n\hat{Ł}^n\hat{Ł}^{*^m}$

$(K\hat{Ł})^n(K\hat{Ł})^{*^m}(K\hat{Ł})^{*^m}(K\hat{Ł})^n - \overline{K}^{*^m}K^nK^n\overline{K}^{*^m}\hat{Ł}^n\hat{Ł}^{*^m}\hat{Ł}^{*^m}\hat{Ł}^n$

$(K\hat{Ł})^n(K\hat{Ł})^{*^m}(K\hat{Ł})^{*^m}(K\hat{Ł})^n - K^n\overline{K}^{*^m}\overline{K}^{*^m}K^n\hat{Ł}^n\hat{Ł}^{*^m}\hat{Ł}^{*^m}\hat{Ł}^n$

$(K\hat{Ł})^n(K\hat{Ł})^{*^m}(K\hat{Ł})^{*^m}(K\hat{Ł})^n - K^n\hat{Ł}^n\overline{K}^{*^m}\hat{Ł}^{*^m}\overline{K}^{*^m}\hat{Ł}^{*^m}K^n\hat{Ł}^n$

$(K\hat{Ł})^n(K\hat{Ł})^{*^m}(K\hat{Ł})^{*^m}(K\hat{Ł})^n - (K\hat{Ł})^n(K\hat{Ł})^{*^m}(K\hat{Ł})^{*^m}(K\hat{Ł})^n = 0$

Then $K\hat{Ł}$ is $[IF-(\mathrm{n,m})-BN]$operator.

3.5. Theorem

IF $\hat{Ł}$ $[FS-(\mathrm{n,m})-BN]$, $B = \hat{Ł}^n\hat{Ł}^{*^m}$, $F = \hat{Ł}^n\hat{Ł}^{*^m} + \hat{Ł}^{*^m}\hat{Ł}^n$ and $G = \hat{Ł}^n\hat{Ł}^{*^m} - \hat{Ł}^{*^m}\hat{Ł}^n$ operator, then B commutes with F and G.

3.5.1. Proof

$$\hat{Ł}^n\hat{Ł}^{*^m}\left(\hat{Ł}^n\hat{Ł}^{*^m} + \hat{Ł}^{*^m}\hat{Ł}^n\right) - \left(\hat{Ł}^n\hat{Ł}^{*^m} + \hat{Ł}^{*^m}\hat{Ł}^n\right)\hat{Ł}^n\hat{Ł}^{*^m}$$

$$\hat{Ł}^n\hat{Ł}^{*^m}\hat{Ł}^n\hat{Ł}^{*^m} + \hat{Ł}^n\hat{Ł}^{*^m}\hat{Ł}^{*^m}\hat{Ł}^n - \hat{Ł}^n\hat{Ł}^{*^m}\hat{Ł}^n\hat{Ł}^{*^m} - \hat{Ł}^{*^m}\hat{Ł}^n\hat{Ł}^n\hat{Ł}^{*^m} = 0$$

Then $\hat{Ł}^n T^{*^m}\left(\hat{Ł}^n\hat{Ł}^{*^m} + \hat{Ł}^{*^m}\hat{Ł}^n\right) - \left(\hat{Ł}^n\hat{Ł}^{*^m} + \hat{Ł}^{*^m}\hat{Ł}^n\right)\hat{Ł}^n\hat{Ł}^{*^m}$

Hence $BF = FB$

$\hat{Ł}^n\, \hat{Ł}^{*^m}(\hat{Ł}^n\, \hat{Ł}^{*^m} - \hat{Ł}^{*^m}\, \hat{Ł}^n) - (\hat{Ł}^n\, \hat{Ł}^{*^m} - \hat{Ł}^{*^m}\, \hat{Ł}^n)\hat{Ł}^n\, \hat{Ł}^{*^m}$

$$\hat{Ł}^n\, \hat{Ł}^{*^m}\hat{Ł}^n\, \hat{Ł}^{*^m} - \hat{Ł}^n\, \hat{Ł}^{*^m}\hat{Ł}^{*^m}\hat{Ł}^n - \hat{Ł}^n\, \hat{Ł}^{*^m}\hat{Ł}^n\, \hat{Ł}^{*^m} + \hat{Ł}^{*^m}\, \hat{Ł}^n\hat{Ł}^n\, \hat{Ł}^{*^m}$$

$$\hat{Ł}^{*^m}\hat{Ł}^n\hat{Ł}^n\, T^{*^m} - \hat{Ł}^n\hat{Ł}^{*^m}\hat{Ł}^n\, \hat{Ł}^{*^m} = 0$$

$$\text{Then } \hat{Ł}^n\, \hat{Ł}^{*^m}(\hat{Ł}^n\, \hat{Ł}^{*^m} - \hat{Ł}^{*^m}\, \hat{Ł}^n) = (\hat{Ł}^n\, \hat{Ł}^{*^m} - \hat{Ł}^{*^m}\, \hat{Ł}^n)\hat{Ł}^n\, \hat{Ł}^{*^m}$$

Hence $BG = GB$.

3.6. Theorem

If $\hat{Ł}$ *and* $\hat{S}$ are $[IF - (\mathrm{n,m}) - BN]$ operators over FSH-space $\widehat{H}$ and doubly commuting then $\hat{Ł}\hat{S}$ is likewise an $[IF - (\mathrm{n,m}) - BN]$ operator**.**

3.6.1. Proof

Show that $\hat{Ł}\,\hat{Ş}$ is an $[IF - (\mathrm{n,m}) - BN]$ operator, we have

$$\begin{aligned}
(\hat{Ł}\hat{Ş}\,)^n(\hat{Ł}\hat{Ş}\,)^{*^m}(\hat{Ł}\hat{Ş}\,)^{*^m}(\hat{Ł}\hat{Ş}\,)^n &= \hat{Ł}^n\hat{Ş}^n\,.\ \hat{Ş}^{*^m}\hat{Ł}^{*^m}.\hat{Ş}^{*^m}\,\hat{Ł}^{*^m}\ .\hat{Ł}^n\hat{Ş}^n \\
&= \hat{Ş}^n\hat{Ş}^{*^m}\ \hat{Ł}^n\,\hat{Ł}^{*^m}\hat{Ş}^{*^m}\hat{Ł}^{*^m}\,\hat{Ł}^n\hat{Ş}^n \\
&= \hat{Ş}^n\,\hat{Ł}^{*^m}\hat{Ş}^{*^m}\ \hat{Ł}^n\hat{Ł}^n\hat{Ş}^{*^m}\hat{Ł}^{*^m}\,\hat{Ş}^n \\
&= \hat{Ł}^{*^m}\hat{Ş}^n\hat{Ł}^n\,\hat{Ş}^{*^m}\,\hat{Ş}^{*^m}\,\hat{Ł}^n\hat{Ş}^n\hat{Ł}^n \\
&= \hat{Ł}^{*^m}\hat{Ł}^n\,\hat{Ş}^n\hat{Ş}^{*^m}\,\hat{Ş}^{*^m}\hat{Ş}^n\hat{Ł}^n\hat{Ł}^{*^m} \\
&= \hat{Ł}^{*^m}\hat{Ł}^n\,\hat{Ş}^{*^m}\hat{Ş}^n\hat{Ş}^n\,\hat{Ş}^{*^m}\ \hat{Ł}^n\hat{Ł}^{*^m} \\
&= \hat{Ł}^{*^m}\,\hat{Ş}^{*^m}\hat{Ł}^n\hat{Ş}^n\hat{Ş}^n\,\hat{Ł}^n\hat{Ş}^{*^m}\hat{Ł}^{*^m} \\
&= \hat{Ş}^{*^m}\hat{Ł}^{*^m}\hat{Ł}^n\hat{Ş}^n\hat{Ł}^n\hat{Ş}^n\,\hat{Ş}^{*^m}\hat{Ł}^{*^m} \\
&= (\hat{Ł}\hat{Ş}\,)^{*^m}(\hat{Ł}\hat{Ş}\,)^n(\hat{Ł}\hat{Ş}\,)^n(\hat{Ł}\hat{Ş}\,)^{*^m}
\end{aligned}$$

Therefore $(\hat{Ł}\hat{S})$ is an $[FS - (\mathrm{n,m}) - BN]$ operator.

3.7. Theorem

If $\hat{Ł}_1, \ldots\ldots, \hat{Ł}_m$ *be* $[FS - (\mathrm{n,m}) - BN]$ operators .Then $(\hat{Ł}_1 \oplus \ldots\ldots \oplus \hat{Ł}_m)$ and $(\hat{Ł}_1 \otimes \ldots\ldots \otimes \hat{Ł}_m)$ are $[FS - (\mathrm{n,m}) - BN]$operators.

3.7.1. Proof

since $\hat{Ł}_1, \ldots\ldots, \hat{Ł}_m$ be $[FS - (\mathrm{n,m}) - BN]$operator

$$(\hat{Ł}_1 \oplus \ldots\ldots \oplus \hat{Ł}_m\,)^n\,(\hat{Ł}_1 \oplus \ldots\ldots \oplus \hat{Ł}_m\,)^{*^m}(\hat{Ł}_1 \oplus \ldots\ldots \oplus \hat{Ł}_m\,)^{*^m}$$

$$(\hat{Ł}_1 \oplus \ldots\ldots \oplus \hat{Ł}_m\,)^n - (\hat{Ł}_1 \oplus \ldots\ldots \oplus \hat{Ł}_m\,)^{*^m}\,(\hat{Ł}_1 \oplus \ldots\ldots \oplus \hat{Ł}_m\,)^n$$

$$(\hat{Ł}_1 \oplus \ldots\ldots \oplus \hat{Ł}_m\,)^n\ (\hat{Ł}_1 \oplus \ldots\ldots \oplus \hat{Ł}_m\,)^{*^m}$$

$$(\,\hat{Ł}_1{}^n \oplus \ldots\ldots \oplus\ \hat{Ł}_m{}^n)\ (\,\hat{Ł}_1{}^{*^m} \oplus\ \ldots\ldots \oplus\ \hat{Ł}_m{}^{*^m}\,)(\,\hat{Ł}_1{}^{*^m} \oplus\ \ldots\ldots \oplus \hat{Ł}_m{}^{*^m})$$

$$(\,\hat{Ł}_1{}^n \oplus \ldots\ldots \oplus \hat{Ł}_m{}^n) - (\,\hat{Ł}_1{}^{*^m} \oplus \ldots\ldots \oplus \hat{Ł}_m{}^{*^m})(\,\hat{Ł}_1{}^n \oplus \ldots\ldots \oplus\ \hat{Ł}_m{}^n)$$

$$(\hat{Ł}_1{}^n \oplus \ldots\ldots \oplus \hat{Ł}_m{}^n)\left(\hat{Ł}_1{}^{*m} \oplus \ldots\ldots \oplus \hat{Ł}_m{}^{*m}\right)$$

$$\left(\hat{Ł}_1{}^n \hat{Ł}_1{}^{*m} \hat{Ł}_1{}^{*m} \hat{Ł}_1{}^n \oplus \ldots\ldots \oplus \hat{Ł}_m{}^n \hat{Ł}_m{}^{*m} \hat{Ł}_m{}^{*m} \hat{Ł}_m{}^n\right)$$

$$-\left(\hat{Ł}_1{}^{*m} \hat{Ł}_1{}^n \hat{Ł}_1{}^n \hat{Ł}_1{}^{*m} \oplus \ldots\ldots \oplus \hat{Ł}_m{}^{*m} \hat{Ł}_m{}^n \hat{Ł}_m{}^n \hat{Ł}_m{}^{*m}\right)$$

since $\hat{Ł}_1, \ldots\ldots, \hat{Ł}_m$ be $[FS-(\mathrm{n},\mathrm{m})-BN]$ operator hence

$$\left(\hat{Ł}_1{}^{*m} \hat{Ł}_1{}^n \hat{Ł}_1{}^n \hat{Ł}_1{}^{*m} \oplus \ldots\ldots \oplus \hat{Ł}_m{}^{*m} \hat{Ł}_m{}^n \hat{Ł}_m{}^n \hat{Ł}_m{}^{*m}\right)$$

$$-\left(\hat{Ł}_1{}^{*m} \hat{Ł}_1{}^n \hat{Ł}_1{}^n \hat{Ł}_1{}^{*m} \oplus \ldots\ldots \oplus \hat{Ł}_m{}^{*m} \hat{Ł}_m{}^n \hat{Ł}_m{}^n \hat{Ł}_m{}^{*m}\right) = 0$$

Then $(\hat{Ł}_1 \oplus \ldots\ldots \oplus \hat{Ł}_m)$ is $[FS-(\mathrm{n},\mathrm{m})-BN]$ operators.

Now, for $x_1, \ldots, x_m \in H$

$$(\hat{Ł}_1 \otimes \ldots\ldots \otimes \hat{Ł}_m)^n (\hat{Ł}_1 \otimes \ldots\ldots \otimes \hat{Ł}_m)^{*m} (\hat{Ł}_1 \otimes \ldots\ldots \otimes \hat{Ł}_m)^{*m}$$

$$(\hat{Ł}_1 \otimes \ldots\ldots \otimes \hat{Ł}_m)^n (x_1 \otimes \ldots \otimes x_m) - (\hat{Ł}_1 \otimes \ldots\ldots \otimes \hat{Ł}_m)^{*m}$$

$$(\hat{Ł}_1 \otimes \ldots\ldots \otimes \hat{Ł}_m)^n (\hat{Ł}_1 \otimes \ldots\ldots \otimes \hat{Ł}_m)^n (\hat{Ł}_1 \otimes \ldots\ldots \otimes \hat{Ł}_m)^{*m} (x_1 \otimes \ldots \otimes x_m)$$

$$(\hat{Ł}_1{}^n \otimes \ldots\ldots \otimes \hat{Ł}_m{}^n)(\hat{Ł}_1{}^{*m} \otimes \ldots\ldots \otimes \hat{Ł}_m{}^{*m})(\hat{Ł}_1{}^{*m} \otimes \ldots\ldots \otimes \hat{Ł}_m{}^{*m})$$

$$(\hat{Ł}_1{}^n \otimes \ldots\ldots \otimes \hat{Ł}_m{}^n)(x_1 \oplus \ldots \otimes x_m) - (\hat{Ł}_1{}^{*m} \otimes \ldots\ldots \otimes \hat{Ł}_m{}^{*m})$$

$$(\hat{Ł}_1{}^n \otimes \ldots\ldots \otimes \hat{Ł}_m{}^n)(\hat{Ł}_1{}^n \otimes \ldots\ldots \otimes \hat{Ł}_m{}^n)(\hat{Ł}_1{}^{*m} \otimes \ldots\ldots \otimes \hat{Ł}_m{}^{*m})(x_1 \otimes \ldots \otimes x_m)$$

$$\left(\hat{Ł}_1{}^n \hat{Ł}_1{}^{*m} \hat{Ł}_1{}^{*m} \hat{Ł}_1{}^n x_1 \otimes \ldots \otimes \hat{Ł}_m{}^n \hat{Ł}_m{}^{*m} \hat{Ł}_m{}^{*m} \hat{Ł}_m{}^n\right)$$

$$-\left(\hat{Ł}_1{}^{*m} \hat{Ł}_1{}^n \hat{Ł}_1{}^n \hat{Ł}_1{}^{*m} x_1 \otimes \ldots \oplus \hat{Ł}_m{}^{*m} \hat{Ł}_m{}^n \hat{Ł}_m{}^n \hat{Ł}_m{}^{*m} x_m\right)$$

Since $\hat{Ł}_1, \ldots\ldots, \hat{Ł}_m$ be $[FS-(\mathrm{n},\mathrm{m})-BN]$operator hence

$$\left(\hat{Ł}_1{}^{*m} \hat{Ł}_1{}^n \hat{Ł}_1{}^n \hat{Ł}_1{}^{*m} x_1 \otimes \ldots \otimes \hat{Ł}_m{}^{*m} \hat{Ł}_m{}^n \hat{Ł}_m{}^n \hat{Ł}_m{}^{*m} x_m\right)$$

$$-\left(\hat{Ł}_1{}^{*m} \hat{Ł}_1{}^n \hat{Ł}_1{}^n \hat{Ł}_1{}^{*m} x_1 \otimes \ldots \otimes \hat{Ł}_m{}^{*m} \hat{Ł}_m{}^n \hat{Ł}_m{}^n \hat{Ł}_m{}^{*m} x_m\right) = 0$$

Thus $(\hat{Ł}_1 \otimes \ldots\ldots \otimes \hat{Ł}_m)$ is $[FS-(\mathrm{n},\mathrm{m})-BN]$ operators.

4. Conclusion

The given research has managed to put forward and systematically explore the notion of the fuzzy soft (n,m) binormal operator in the concept of the fuzzy soft Hilbert spaces. We have applied classical binormal operator theory to the fuzzy soft environment, and have thereby provided a rigorous framework to the new type of operator, as well as clarified its basic structural properties. The major contributions of the work are:

The key contributions of this work include

- The exact definition of fuzzy soft (n,m) binormal operators, in terms of the commutation relations which define them in terms of powers of the operator and powers of its adjoint.
- The fact that the adjoint of an FS-(n,m)-BN operator is an FS-(m,n)-BN operator, which can be said to be a duality principle.
- Scalar multiplication Always has the (n,m) binormal property Fine-Tuning The proof that the (n,m) binormal property is preserved under scalar multiplication.
- The establishment of important commutativity relationships between the operator $B = L^nL^m$ and the combinations $F = L^nL^m + L^mL^n$ and $G = L^nL^m - L^{*m}L^n$.
- The fact that the (n,m) binormal property is product-invariant, direct sum-invariant and tensor product-invariant of doubly commuting operators.

The findings do not only add variety to the theoretical background of fuzzy soft operator theory, but also give the necessary means to further inquiry into the spectral characteristics and structural categorization of operators in fuzzy soft Hilbert spaces. The interaction between the fuzzy set theory, the soft set theory, and the classical operator theory presented in this work gives new opportunities to face the uncertainty in functional analytic frameworks.

Compliance with ethical standards

Disclosure of conflict of interest

No conflict of interest to be disclosed.

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